Crystallizing compact semisimple Lie groups

Robert Yuncken
(joint work with Marco Matassa)

IECL Metz

20 November 2023

Executive summary

1. **Goal:** "Set-theoretic" version of representation theory for semisimple Lie groups.
2. This is well-known for compact s.s. Lie groups K: **Crystal bases**.
3. This talk is about crystalization for the AN group.

Crystal bases

Irred. representations of a compact semisimple Lie group

K — compact semisimple Lie group (connected, simply connected)

$\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ — complexification of its Lie algebra

$\Longrightarrow$ irred. unitary rep'ns of $K \equiv$ irred. rep'ns of $\mathfrak{g}$.

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, generated by

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & \ddots & \\ & 1 & & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

Irred. representations of a compact semisimple Lie group

K — compact semisimple Lie group (connected, simply connected)

$\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ — complexification of its Lie algebra
$\implies$ irred. unitary rep'ns of $K \equiv$ irred. rep'ns of $\mathfrak{g}$.

Theorem

Irred. rep'ns of $\mathfrak{g}$ are classified by highest weight $\lambda \in \mathbf{P}^+$.

Explicit structure. . . ?

Irred. representations of a compact semisimple Lie group

K — compact semisimple Lie group (connected, simply connected)

$\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ — complexification of its Lie algebra
$\implies$ irred. unitary rep'ns of K $\equiv$ irred. rep'ns of $\mathfrak{g}$.

Theorem

Irred. rep'ns of $\mathfrak{g}$ are classified by highest weight $\lambda \in \mathbf{P}^+$.

Explicit structure. . . ?

- **Weyl character formula (1925):** Action of Cartan subalgebra $\mathfrak{h}$.
- **Gelfand-Tsetlin (1950):** Explicit formulas for action of simple root vectors E_i, F_i, but for $\mathfrak{gl}_n(\mathbb{C})$ only.
- **Pand-Hecht, Wong (1967), & many others:** Same for $\mathfrak{o}_n$, then $\mathfrak{sp}_n$, then all classical $\mathfrak{g}$.
- **Kashiwara, Lusztig (1990):** Crystal bases (asymptotic formulas + much more)

Example: $V(2\varpi_1 + 2\varpi_2)$

REPN OF $\mathcal{U}_q(\mathfrak{sl}_3)$ WITH HIGHEST WEIGHT (2,0,-2)

28 September 2011

Given w.r.t. orthonormal Gelfand-Tsetlin basis

Notes

- All bases ordered by descending highest $\mathcal{U}_q(\mathfrak{sl}_2)$-weights.
- All numbers are q-numbers.

  $\longleftarrow = F_1$ (LEFT)

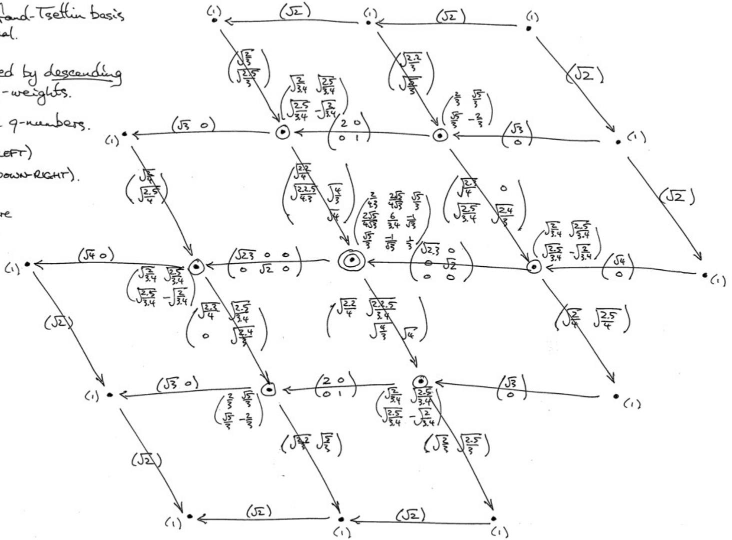
  $\longleftarrow = F_2$ (DOWN-RIGHT).

- Black matrices are change of basis.

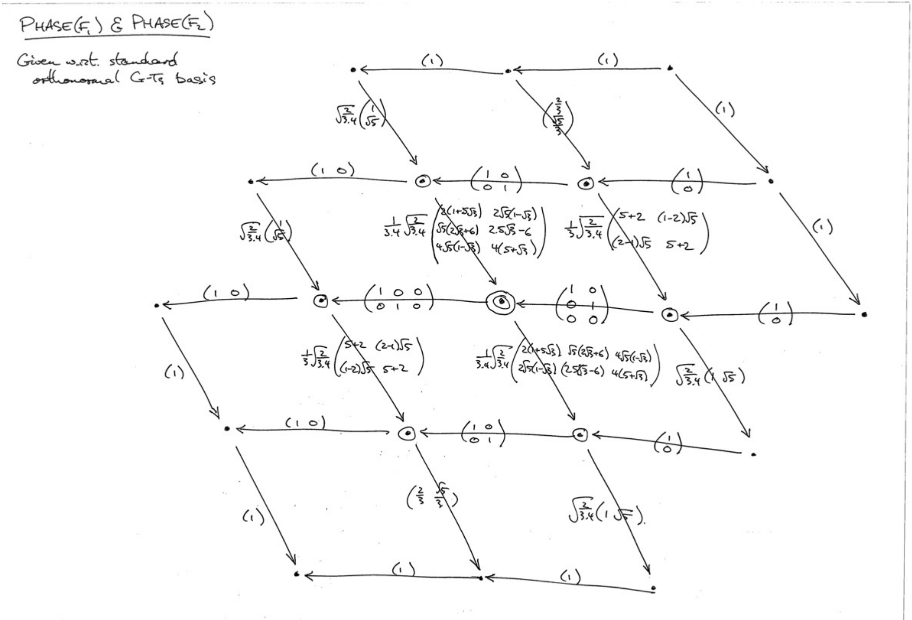
  IN: Coords w.r.t. alternative G-Ts basis.

  OUT: Coords w.r.t. standard G-Ts basis.

  i.e. Columns denote alternative G-Ts basis vectors in standard G-Ts basis.



Example: $V(2\varpi_1 + 2\varpi_2)$ for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$



Quantized envelopping algebras (Drinfeld-Jimbo)

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$,

$U(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & 1 & \ddots & \\ & & & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

with the Chevalley-Serre relations:

$$[H_j, E_i] = \alpha_i(H_j) E_i \qquad \alpha_i : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} \mapsto a_i - a_{i+1}$$

$$[H_j, F_i] = -\alpha_i(H_j) F_i$$

$$[E_i, F_j] = \delta_{ij} H_i$$

$$E_i^2 E_{i\pm1} - 2E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0$$

$$F_i^2 F_{i\pm1} - 2F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

Quantized envelopping algebras (Drinfeld-Jimbo)

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, $q \in \mathbb{R}_+^\times$, $q \neq 1$

$U_q(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & 1 & \ddots & \\ & & & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

with the Chevalley-Serre relations:

$$[H_j, E_i] = \alpha_i(H_j)E_i$$

$$[H_j, F_i] = -\alpha_i(H_j)F_i$$

$$\alpha_i : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} \mapsto a_i - a_{i+1}$$

$$[E_i, F_j] = \delta_{ij}[H_i]_q$$

$$E_i^2 E_{i\pm1} - [2]_q E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0$$

$$F_i^2 F_{i\pm1} - [2]_q F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

where: $[2]_q = q + q^{-1}$, $[H]_q = \frac{q^H - q^{-H}}{q - q^{-1}}$.

Rmk. Actually, one uses $K_i = q^{H_i}$ instead of H_i as generators.

Example: $V(2\varpi_1 + 2\varpi_2)$

REPN OF $\mathcal{U}_q(\mathfrak{sl}_3)$ WITH HIGHEST WEIGHT $(2,0,-2)$

28 September 2011

Given w.r.t. orthonormal Gelfand-Tsetlin basis

Notes

- All bases ordered by descending highest $\mathcal{U}_q(\mathfrak{sl}_2)$-weights.
- All numbers are q-numbers.

  $\longleftarrow = F_1$ (LEFT)

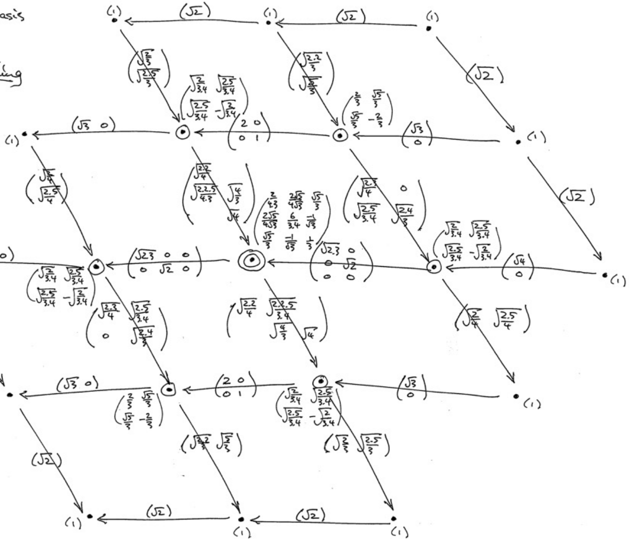
  $\swarrow = F_2$ (DOWN-RIGHT).

- Black matrices are change of basis.

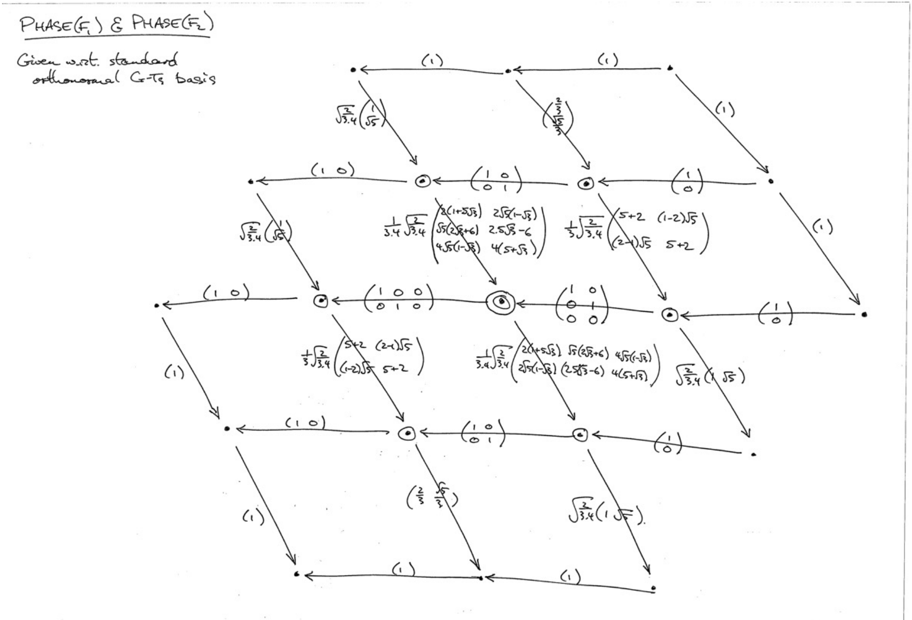
  IN: Coords w.r.t. alternative G-Ts basis.

  OUT: Coords w.r.t. standard G-Ts basis.

  i.e. Columns denote alternative G-Ts basis vectors in standard G-Ts basis.

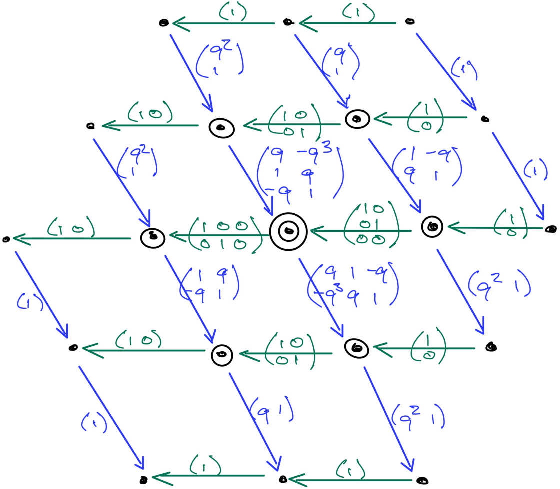


Example: $V(2\varpi_1 + 2\varpi_2)$



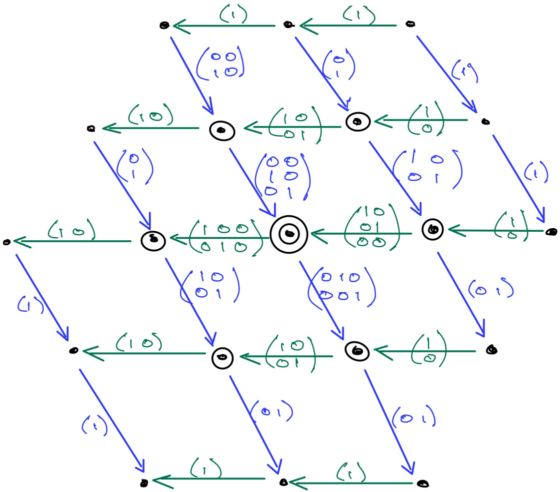
Example: $V(2\varpi_1 + 2\varpi_2)$

DOMINANT TERMS
IN PH(F_1) & PH(F_2)
AS $q \to 0$



Example: $V(2\varpi_1 + 2\varpi_2)$

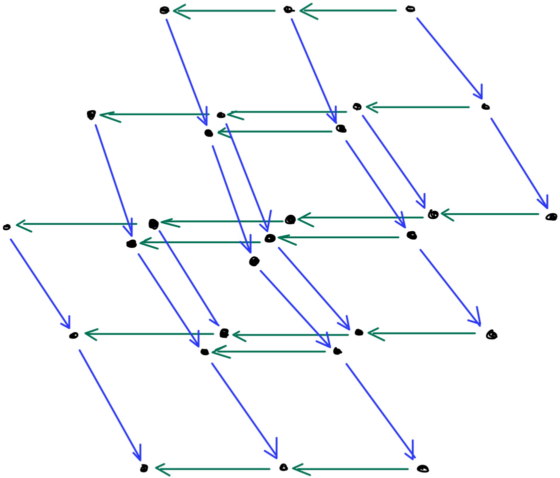
LIMIT AS $q \to 0$



Example: $V(2\varpi_1 + 2\varpi_2)$

LIMIT AS $q \to 0$

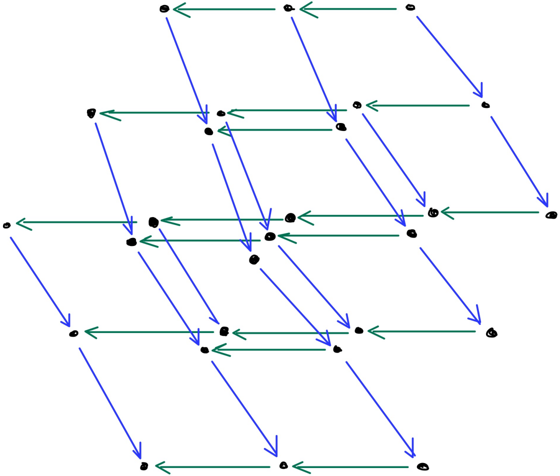
(CRYSTAL GRAPH)



Example: $\mathcal{B}(2\varpi_1 + 2\varpi_2)$

LIMIT AS $q \to 0$

(CRYSTAL GRAPH)



Tensor product of crystals

The crystal limit ($q \to 0$) simplifies not just the action of the generators, but also the Clebsch-Gordan coefficients, branching rules, ...

Tensor product of crystals

The crystal limit ($q \to 0$) simplifies not just the action of the generators, but also the Clebsch-Gordan coefficients, branching rules, . . .

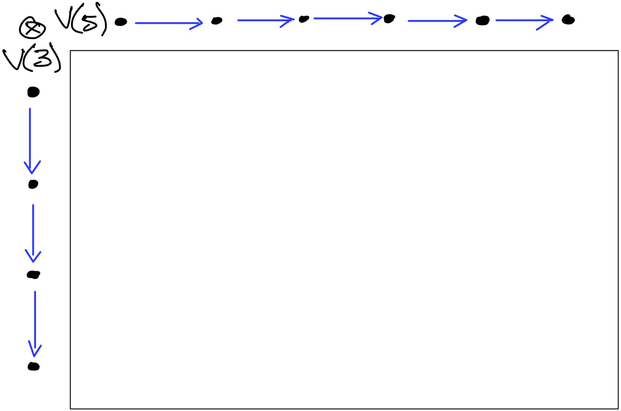
Theorem (Tensor product rule)

If $(\mathcal{L}, \mathcal{B})$, $(\mathcal{L}', \mathcal{B}')$ are crystal bases for V, V', then $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \times \mathcal{B}')$ is a crystal basis for $V \otimes V'$, with action

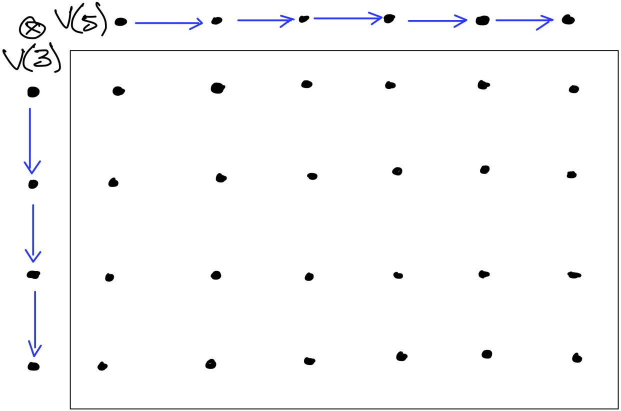
$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

where $\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\}$ and $\varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}$.

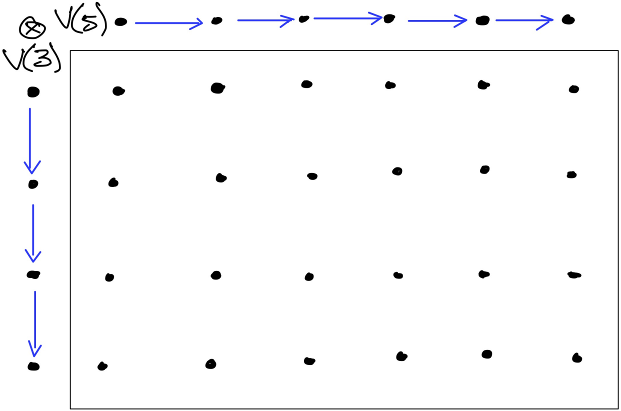
Example: Tensor product of $\mathfrak{sl}_2$ representations



Example: Tensor product of $\mathfrak{sl}_2$ representations

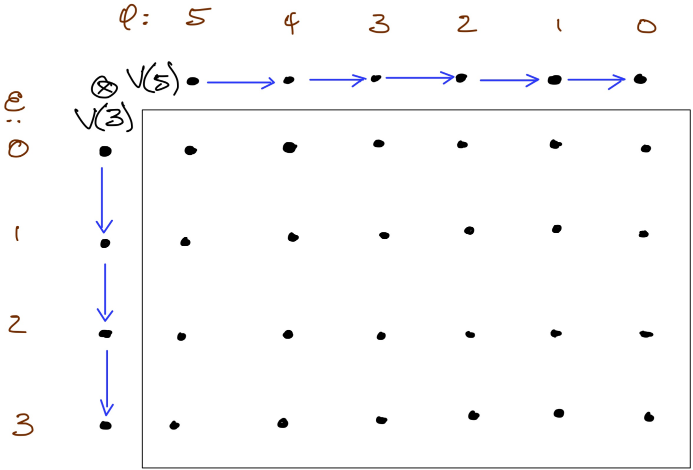


Example: Tensor product of $\mathfrak{sl}_2$ representations



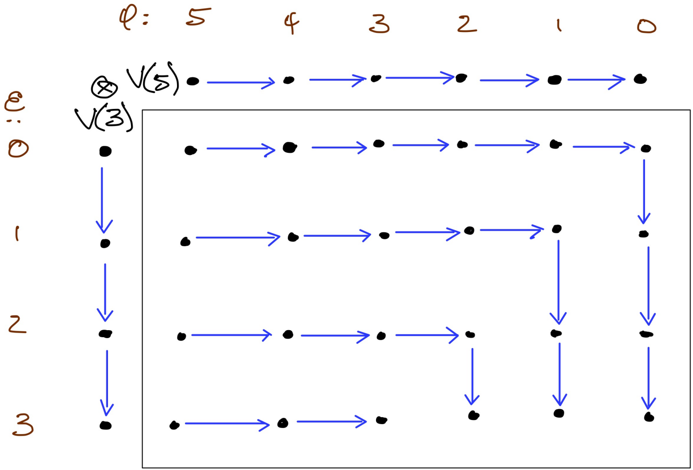
$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

Example: Tensor product of $\mathfrak{sl}_2$ representations



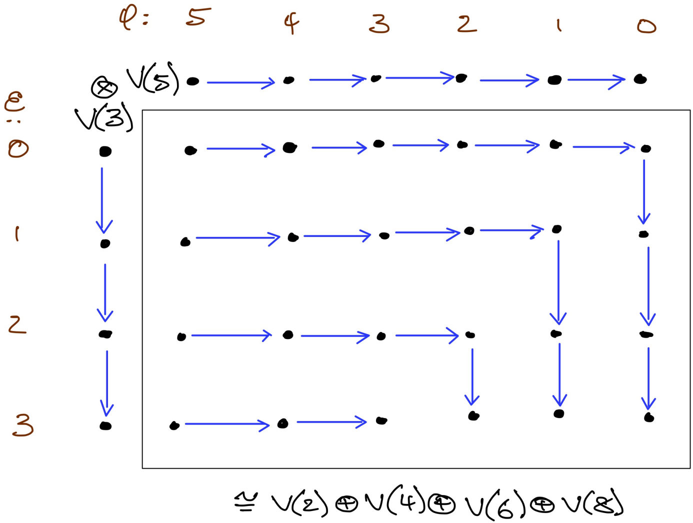
$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

Example: Tensor product of $\mathfrak{sl}_2$ representations

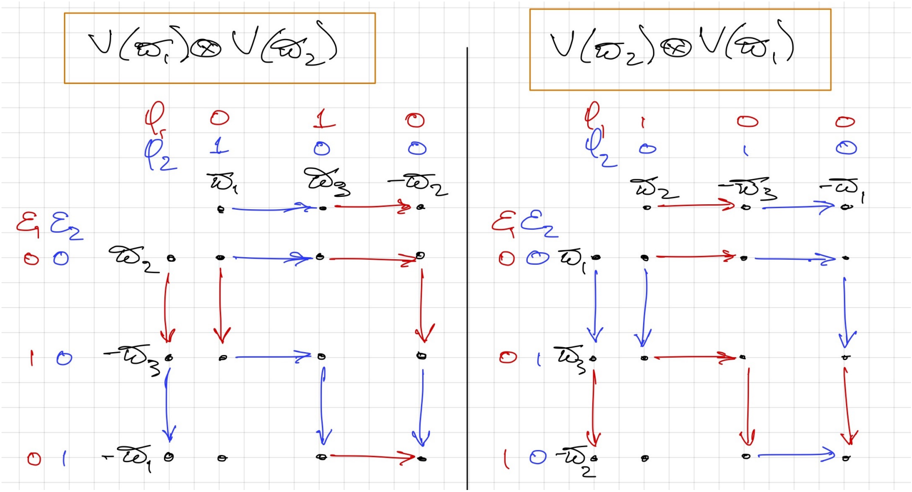


$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

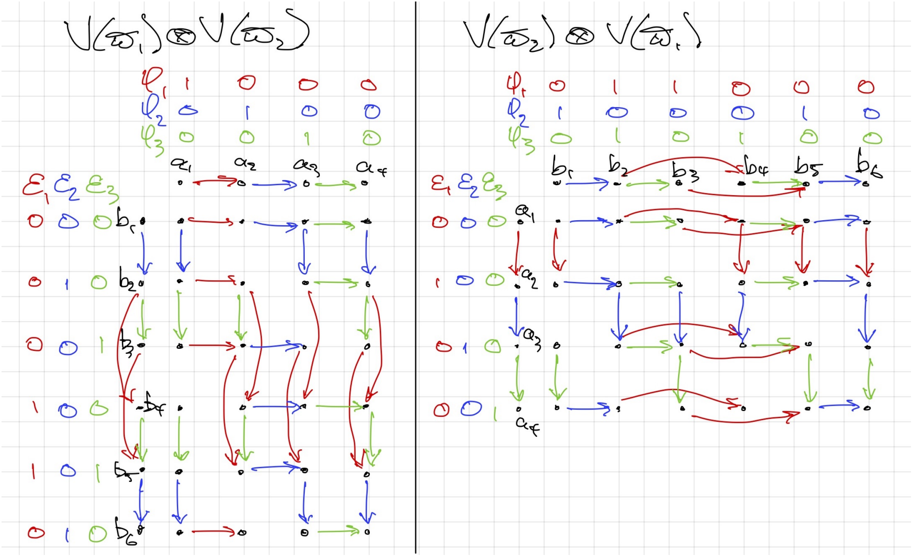
Example: Tensor product of $\mathfrak{sl}_2$ representations



Example: Tensor product of $\mathfrak{sl}_3$ representations



Example: Tensor product of $\mathfrak{sl}_4$ representations



"Crystallization" in analysis: Quantized algebras of functions

Quantized algebras of functions

$$\begin{aligned}\mathcal{O}(K) &= \{\text{polynomial functions on } K\} \\ &= \{\langle\xi| \cdot |\eta\rangle \mid \xi,\eta \in V \text{ (irred. integrable } \mathfrak{g}\text{-modules)}\} \\ C(K) &= \overline{\mathcal{O}(K)}^{\|\cdot\|} \qquad \text{— } C^*\text{-closure}\end{aligned}$$

Quantized algebras of functions

$$\mathcal{O}(K_q) = \{\text{polynomial functions on } K_q\}$$
$$= \{\langle\xi|\ \cdot\ |\eta\rangle \mid \xi, \eta \in V \text{ (irred. integrable } \mathcal{U}_q(\mathfrak{g})\text{-modules)}\}$$

$$C(K_q) = \overline{\mathcal{O}(K_q)}^{\|\cdot\|} \quad \text{— } C^*\text{-closure}$$

Quantized algebras of functions

$$\mathcal{O}(K_q) = \{\text{polynomial functions on } K_q\}$$
$$= \{\langle\xi| \cdot |\eta\rangle \,|\, \xi, \eta \in V \text{ (irred. integrable } \mathcal{U}_q(\mathfrak{g})\text{-modules)}\}$$

$$C(K_q) = \overline{\mathcal{O}(K_q)}^{\|\cdot\|} \quad \text{— } C^*\text{-closure}$$

More generally, we can define $\mathcal{O}(X_q)$ and $C(X_q)$ for any $X = K/H$ with H a Poisson subgroup.

Quantized algebras of functions

$$\mathcal{O}(K_q) = \{\text{polynomial functions on } K_q\}$$
$$= \{\langle\xi|\cdot|\eta\rangle \,|\, \xi,\eta \in V \text{ (irred. integrable } \mathcal{U}_q(\mathfrak{g})\text{-modules)}\}$$

$$C(K_q) = \overline{\mathcal{O}(K_q)}^{\|\cdot\|} \quad \text{— } C^*\text{-closure}$$

More generally, we can define $\mathcal{O}(X_q)$ and $C(X_q)$ for any $X = K/H$ with H a Poisson subgroup.

Remarks.

- **. . . , Neshveyev-Tuset (2012):** The algebras $C(K_q)$ form a continuous field of C^*-algebras for $0 < q < \infty$.
- **. . . , Giselsson (2023):** $C(K_q)$ are all isomorphic for $q \in (0,\infty)\setminus\{1\}$.
- The $\mathcal{O}(K_q)$ are not, though.

Quantized algebras of functions: $q = 0$ limit

Theorem (Woronowicz '87, Hong-Szymański '02, Giselsson '23)

For $X_q = \mathrm{SU}_q(2)$, $\mathbb{C}P^n_q$, $\mathrm{SU}_q(3)$, the continuous field $(C(X_q))$ extends to $q = 0, \infty$.
All fibres for $q \neq 1$ are isomorphic, and they are **graph C^*-algebras**.

Quantized algebras of functions: $q = 0$ limit

Theorem (Woronowicz '87, Hong-Szymański '02, Giselsson '23)

For $X_q = \mathrm{SU}_q(2)$, $\mathbb{C}P^n_q$, $\mathrm{SU}_q(3)$, the continuous field $(C(X_q))$ extends to $q = 0, \infty$.
All fibres for $q \neq 1$ are isomorphic, and they are **graph C^*-algebras**.

Remarks.

- Again, the $\mathcal{O}(X_q)$ are not all isomorphic.
- $\mathcal{O}(X_0)$ is a **Leavitt path algebra**
  (= algebraic analog of a graph C^*-algebra).
- More precisely:
  - $C(\mathbb{C}P^n_q)$ is the AF-core of a graph C^*-algebra,
  - $C(\mathrm{SU}_q(3))$ is a higher-rank graph C^*-algebra.
- For $SU_q(n)$, there is another approach to crystallization by Giri-Pal (2023).

Graph algebras

(Λ^0, Λ^1) — directed graph.

Write $\Lambda = \{\text{paths in the graph}\}$

Graph algebras

(Λ^0, Λ^1) — directed graph.

Write $\Lambda = \{\text{paths in the graph}\}$

NB. It's a category with length function $\mathbf{d} : \Lambda \to \mathbb{N}$.

Graph algebras

(Λ^0, Λ^1) — directed graph.

Write $\Lambda = \{\text{paths in the graph}\}$

NB. It's a category with length function $\mathbf{d} : \Lambda \to \mathbb{N}$.

Definition

The **Leavitt path algebra** $\mathrm{KP}^*(\Lambda)$ is the universal $*$-algebra generated by projections p_v $(v \in \Lambda^0)$ and partial isometries s_e $(e \in \Lambda^1)$ satisfying:

1. p_v are mutually orthogonal projections
2. $s_e^* s_{e'} = \delta_{ee'} p_{s(e)}$
3. $p_v = \sum_{r(e)=v} s_e s_e^*$

The **graph** C^***-algebra** $C^*(\Lambda)$ is the enveloping C^*-algebra.

Examples

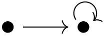
Examples

Classic examples

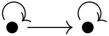
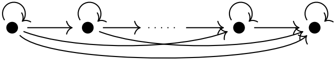
- $C^*(\,\bullet\!\circlearrowleft\,) \cong C(\mathbb{T})$
- $C^*(\,\bullet \longrightarrow \bullet\!\circlearrowleft\,) \cong \mathcal{T}$ (Toeplitz alg.)

Examples

Classic examples

- $C^*($  $) \cong C(\mathbb{T})$
- $C^*($  $) \cong \mathcal{T}$ (Toeplitz alg.)

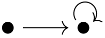
Quantized function algebras $(q \neq 1)$

- $C^*($  $) \cong C(\mathrm{SU}_q(2))$ —Woronowicz
- $C^*($  $) \cong C(Y_q)$ — Hong-Szymański

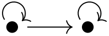
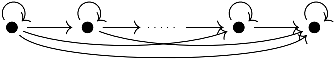
  Y = canonical $\mathbb{T}$-bundle over $\mathbb{C}P^n$

Examples

Classic examples

- $C^*($  $) \cong C(\mathbb{T})$
- $C^*($  $) \cong \mathcal{T}$ (Toeplitz alg.)

Quantized function algebras $(q \neq 1)$

- $C^*($  $) \cong C(\mathrm{SU}_q(2))$ —Woronowicz
- $C^*($  $) \cong C(Y_q)$ — Hong-Szymański

  Y = canonical $\mathbb{T}$-bundle over $\mathbb{C}P^n$

Question: What are these graphs?
What about higher rank?

Higher rank graphs

Definition (k-graph)

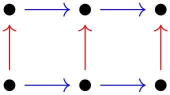
A k-**graph** is a category Λ (of paths) with a morphism $\mathbf{d} : \Lambda \to \mathbb{N}^k$ (*degree* or *coloured length*) satisfying the *factorization property*:

$$\mathbf{d}(e) = m + n \quad \Longrightarrow \quad e = e_1 e_2 \text{ uniquely with } \mathbf{d}(e_1) = m,\ \mathbf{d}(e_2) = n.$$

Notation:

- $\Lambda^n := \mathbf{d}^{-1}(n)$ — paths of coloured length $n \in \mathbb{N}^k$.
- $\Lambda^{(0,\ldots,0)}$ is the set of **vertices**.
- $\Lambda^{(0,\ldots,1,\ldots,0)}$ is the set of **edges of colour** i.

eg,



Higher rank graph algebras

Definition

The **Kumjian-Pask algebra** $\mathrm{KP}^*(\Lambda)$ of a k-graph Λ is the universal $*$-algebra generated by projections p_v $(v \in \Lambda^0)$ and partial isometries s_e $(e \in \Lambda^{\neq 0})$ satisfying $s_{ee'} = s_e s_{e'}$ and:

1. p_v are mutually orthogonal projections.
2. $s_e^* s_{e'} = \delta_{ee'} p_{s(e)}$ when $\mathbf{d}(e) = \mathbf{d}(e')$.
3. $p_v = \sum_{\substack{r(e)=v \\ \mathbf{d}(e)=n}} s_e s_e^*$ for any $n \neq 0$ fixed.

The **higher rank graph** C^***-algebra** $C^*(\Lambda)$ is the enveloping C^*-algebra.

Crystallized algebras of functions

Theorem (Matassa-Y.)

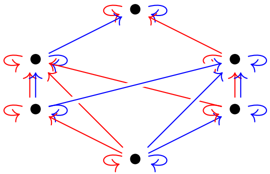
K — connected compact semisimple Lie group.
The continuous field of C^*-algebras $(C(K_q))_{q\in(0,\infty)}$ extends to $q = 0, \infty$ with $C(K_0) \cong C(K_\infty) \cong C^*(\Lambda)$ for some explicit **higher rank graph** *Λ of rank $\mathrm{rk}(K)$.*

Remarks

- For $K = \mathrm{SU}(3)$, this is a result of Giselsson.
  He also shows $C(\mathrm{SU}_q(3)) \cong C^*(\Lambda)$ for every $q \in [0, \infty]\setminus\{1\}$.
  Conjecture: This is true in generality.
- Can replace K by the canonical torus bundle Y over a flag manifold.
  The algebra of functions on the flag manifold itself is the gauge-invariant subalgebra. At $q = 0, \infty$, this is the AF-core.

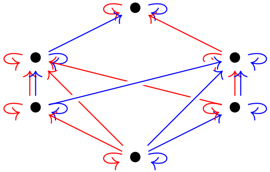
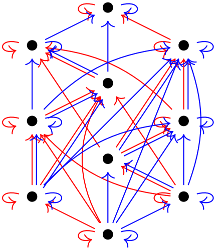
Crystallized algebras of functions

Examples

- $C(SU_0(3)) \cong C^*(\Lambda)$, where $\Lambda =$ 

Crystallized algebras of functions

Examples

- $C(SU_0(3)) \cong C^*(\Lambda)$, where $\Lambda =$ 
- $C(Sp_q(2)) \cong C^*(\Lambda)$ at $q = 0$, where $\Lambda =$ 

Graph algebras from crystals

The higher rank graph of a complex semisimple group

For $\lambda \in \mathbf{P}^+$, let $\mathcal{B}(\lambda)$ = crystal graph of $V(\lambda)$.

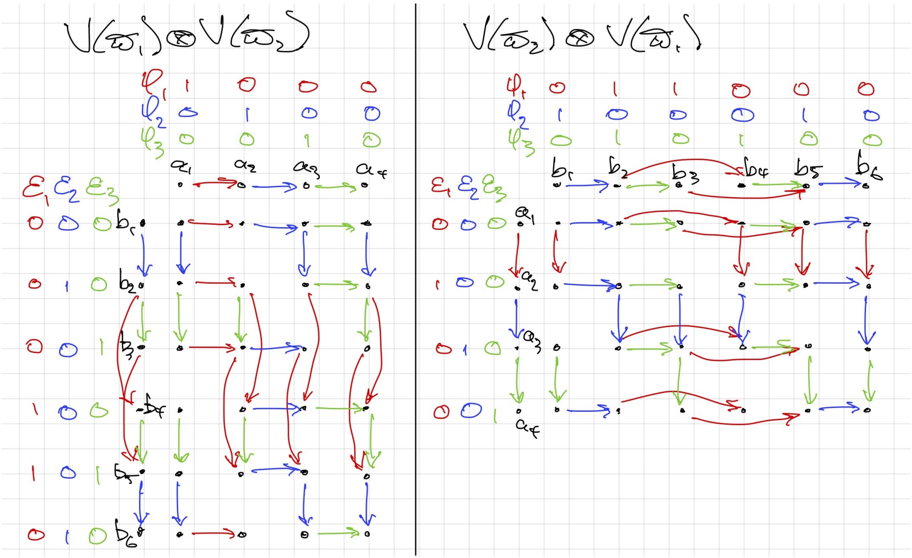
The higher rank graph of a complex semisimple group

For $\lambda \in \mathbf{P}^+$, let $\mathcal{B}(\lambda)$ = crystal graph of $V(\lambda)$.

Definition

- The **Cartan component** of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is the irreducible component of highest weight $\lambda + \mu$.
- Say $b \in \mathcal{B}(\lambda)$, $c \in \mathcal{B}(\mu)$ are **composable** if $b \otimes c \in$ Cartan component.

Example: Tensor product of $\mathfrak{sl}_4$ representations



Right ends of a simple module

Let $\lambda \in \mathbf{P}^+$.
If $\nu \leqslant \lambda$ then we have a tensor product decomposition

$$\mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda - \nu) \otimes \mathcal{B}(\nu)$$

Right ends of a simple module

Let $\lambda \in \mathbf{P}^+$.
If $\nu \leqslant \lambda$ then we have a tensor product decomposition

$$\mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda - \nu) \otimes \mathcal{B}(\nu)$$

Definition (ν-right end)

$$\mathbf{R}_\nu : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda - \nu) \otimes \mathcal{B}(\nu) \to \mathcal{B}(\nu); \quad b \hookrightarrow b' \otimes b'' \mapsto b''$$

Right ends of a simple module

Let $\lambda \in \mathbf{P}^+$.
If $\nu \leqslant \lambda$ then we have a tensor product decomposition

$$\mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda - \nu) \otimes \mathcal{B}(\nu)$$

Definition (ν-right end)

$$\mathbf{R}_\nu : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\lambda - \nu) \otimes \mathcal{B}(\nu) \to \mathcal{B}(\nu); \quad b \hookrightarrow b' \otimes b'' \mapsto b''$$

Lemma

Let $b \in \mathcal{B}(\lambda)$, $c \in \mathcal{B}(\mu)$ with $\lambda \geqslant \rho$. Then b and c are composable iff the fundamental right ends $\mathbf{R}_{\varpi_i}(b)$ are composable with c for every i.

The higher rank graph of a complex semisimple group

Put $r = \mathbf{rk}(K)$, so that $\mathbf{P}^+ \cong \mathbb{N}^r$.

Define an r-graph Λ_K by:

- $\Lambda_K^0 := \{(\mathbf{R}_{\varpi_1}(b), \ldots, \mathbf{R}_{\varpi_r}(b)) \mid b \in \mathcal{B}(\rho)\}$
  — families of fund. right ends of $\mathcal{B}(\rho)$.
- $\Lambda_K^\lambda := \{(v, b) \mid v \in \Lambda^0,\ b \in \mathcal{B}(\lambda) \text{ composable}\}$.

The higher rank graph of a complex semisimple group

Put $r = \mathbf{rk}(K)$, so that $\mathbf{P}^+ \cong \mathbb{N}^r$.

Define an r-graph Λ_K by:

- $\Lambda_K^0 := \{(\mathbf{R}_{\varpi_1}(b), \ldots, \mathbf{R}_{\varpi_r}(b)) \mid b \in \mathcal{B}(\rho)\}$
  — families of fund. right ends of $\mathcal{B}(\rho)$.
- $\Lambda_K^\lambda := \{(v, b) \mid v \in \Lambda^0,\ b \in \mathcal{B}(\lambda) \text{ composable}\}$.

Theorem (Matassa-Y.)

$\mathcal{O}(K_0) \cong \mathrm{KP}(\Lambda_K)$ and $C(K_0) \cong C^*(\Lambda_K)$.

Generators for $C(K_q)$

$v_0^\lambda, \ldots, v_n^\lambda \in V(\lambda)$ —weight basis lifting $b_0^\lambda, \ldots, b_n^\lambda$. ($v_0^\lambda$ = highest)
$f_\lambda^0, \ldots, f_\lambda^n \in V(\lambda)^*$ —dual basis.

Put $\mathbf{f}_i^\lambda = \langle f_\lambda^i| \cdot |v_0^\lambda\rangle$ $\mathbf{v}_i^\lambda = S(\langle f_\lambda^0| \cdot |v_i^\lambda\rangle)$.

Generators for $C(K_q)$

$v_0^\lambda, \ldots, v_n^\lambda \in V(\lambda)$ —weight basis lifting $b_0^\lambda, \ldots, b_n^\lambda$. ($v_0^\lambda$ = highest)
$f_\lambda^0, \ldots, f_\lambda^n \in V(\lambda)^*$ —dual basis.

Put $\mathbf{f}_i^\lambda = \langle f_\lambda^i | \cdot | v_0^\lambda \rangle$ $\mathbf{v}_i^\lambda = S(\langle f_\lambda^0 | \cdot | v_i^\lambda \rangle)$.

Theorem (Stokman '03)

$\mathcal{O}[K_q]$ is generated as a $\mathbb{C}$-algebra by $\mathbf{f}_i^\lambda$, $\mathbf{v}_i^\lambda$, for all $0 < q < \infty$.

Generators for $C(K_q)$

$v_0^\lambda, \ldots, v_n^\lambda \in V(\lambda)$ —weight basis lifting $b_0^\lambda, \ldots, b_n^\lambda$. ($v_0^\lambda$ = highest)
$f_\lambda^0, \ldots, f_\lambda^n \in V(\lambda)^*$ —dual basis.

Put $\mathbf{f}_i^\lambda = \langle f_\lambda^i | \cdot | v_0^\lambda \rangle$ $\mathbf{v}_i^\lambda = S(\langle f_\lambda^0 | \cdot | v_i^\lambda \rangle)$.

Theorem (Stokman '03)

$\mathcal{O}[K_q]$ is generated as a $\mathbb{C}$-algebra by $\mathbf{f}_i^\lambda$, $\mathbf{v}_i^\lambda$, for all $0 < q < \infty$.

Definition. $\mathcal{O}_q^{\mathcal{A}_0}[K] :=$ $\mathcal{A}_0$-algebra generated by $\mathbf{f}_i^\lambda$, $\mathbf{v}_i^\lambda$.

NB. $\mathcal{O}_q^{\mathcal{A}_0}[K] \neq \mathcal{O}_q^{\mathcal{A}_0}[G]$ studied by Kashiwara, Iglesias, which is not $*$-stable.

Generators for $C(K_q)$

$v_0^\lambda, \dots, v_n^\lambda \in V(\lambda)$ —weight basis lifting $b_0^\lambda, \dots, b_n^\lambda$. ($v_0^\lambda$ = highest)
$f_\lambda^0, \dots, f_\lambda^n \in V(\lambda)^*$ —dual basis.

Put $\mathbf{f}_i^\lambda = \langle f_\lambda^i | \cdot | v_0^\lambda \rangle \qquad \mathbf{v}_i^\lambda = S(\langle f_\lambda^0 | \cdot | v_i^\lambda \rangle).$

Theorem (Stokman '03)

$\mathcal{O}[K_q]$ is generated as a $\mathbb{C}$-algebra by $\mathbf{f}_i^\lambda$, $\mathbf{v}_i^\lambda$, for all $0 < q < \infty$.

Definition. $\mathcal{O}_q^{\mathcal{A}_0}[K] :=$ $\mathcal{A}_0$-algebra generated by $\mathbf{f}_i^\lambda$, $\mathbf{v}_i^\lambda$.

NB. $\mathcal{O}_q^{\mathcal{A}_0}[K] \neq \mathcal{O}_q^{\mathcal{A}_0}[G]$ studied by Kashiwara, Iglesias, which is not $*$-stable.

Theorem (Matassa-Y.)

For any $a \in \mathcal{O}_q^{\mathcal{A}_0}[K]$, the operators $\pi_q(a)$ admit a norm limit as $q \to 0$, where $\pi_q : \mathcal{O}[K_q] \to \ell^2(\mathbb{N})^{\otimes k}$ is any Soibelman $*$-representations (corresp. to symplectic leaves).

Relations for $C(K_q)$

$$\langle f^i_\lambda| \cdot |v^\lambda_j\rangle . \langle f^k_\mu| \cdot |v^\mu_l\rangle = \langle f^i_\lambda \otimes f^k_\mu| \cdot |v^\lambda_j \otimes v^\mu_l\rangle.$$

Relations for $C(K_q)$

$$\langle f^i_\lambda | \cdot | v^\lambda_j \rangle . \langle f^k_\mu | \cdot | v^\mu_l \rangle = \langle f^i_\lambda \otimes f^k_\mu | \cdot | v^\lambda_j \otimes v^\mu_l \rangle.$$

NB: $V(\lambda) \otimes V(\mu) \cong V(\mu) \otimes V(\lambda)$, but not via the flip map.

Relations for $C(K_q)$

$$\langle f^i_\lambda| \cdot |v^\lambda_j\rangle . \langle f^k_\mu| \cdot |v^\mu_l\rangle = \langle f^i_\lambda \otimes f^k_\mu| \cdot |v^\lambda_j \otimes v^\mu_l\rangle.$$

NB: $V(\lambda) \otimes V(\mu) \cong V(\mu) \otimes V(\lambda)$, but not via the flip map.

Braiding operators

The integrable $\mathcal{U}_q(\mathfrak{g})$-modules form a **braided category**.
Thus we have operators $\hat{R}_{VW} : V \otimes W \to W \otimes V$ satisfying the **braid relations**

$$U\otimes V\otimes W \xrightarrow{\hat{R}_{UV}\otimes 1} V\otimes U\otimes W \xrightarrow{1\otimes\hat{R}_{UW}} V\otimes W\otimes U \xrightarrow{\hat{R}_{VW}\otimes 1} W\otimes V\otimes U$$

$$U\otimes V\otimes W \xrightarrow{1\otimes\hat{R}_{VW}} U\otimes W\otimes V \xrightarrow{\hat{R}_{UW}\otimes 1} W\otimes U\otimes V \xrightarrow{1\otimes\hat{R}_{UV}} W\otimes V\otimes U$$

Relations for $C(K_q)$

$$\langle f^i_\lambda| \cdot |v^\lambda_j\rangle . \langle f^k_\mu| \cdot |v^\mu_l\rangle = \langle f^i_\lambda \otimes f^k_\mu| \cdot |v^\lambda_j \otimes v^\mu_l\rangle .$$

NB: $V(\lambda) \otimes V(\mu) \cong V(\mu) \otimes V(\lambda)$, but not via the flip map.

Lemma

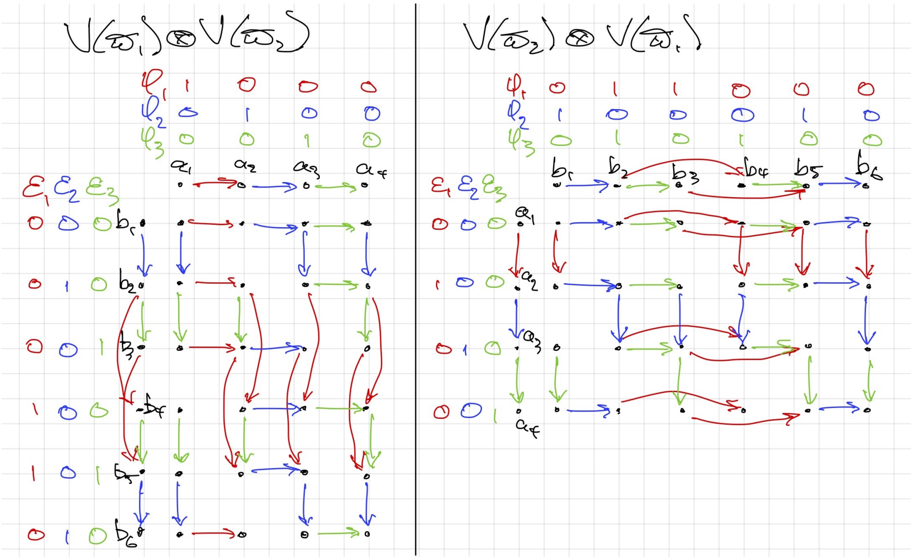
The renormalized braiding ops $q^{(\lambda,\mu)} \hat{R}_{V(\lambda),V(\mu)}$ *descend to crystal morphisms*

$$\sigma_{\mathcal{B}(\lambda),\mathcal{B}(\mu)} : \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \to \mathcal{B}(\mu) \otimes \mathcal{B}(\lambda)$$

which are $\cong$ *on the* **Cartan component** *and zero on other components.*

Let's call this the **Cartan braiding**.

Example: Tensor product of $\mathfrak{sl}_4$ representations



Relations for $C(K_q)$

Corollary

The following relations hold in the crystal limit $\mathcal{O}[K_0]$*:*

1. $\mathbf{f}_i^\lambda \mathbf{f}_j^\mu = \eta(b_i^\lambda \otimes b_j^\mu)\mathbf{f}_m^{\lambda+\mu}, \qquad \mathbf{v}_j^\mu \mathbf{v}_i^\lambda = \eta(b_i^\lambda \otimes b_j^\mu)\mathbf{v}_m^{\lambda+\mu},$

   where $b_i^\lambda \otimes b_j^\mu \mapsto b_m^{\lambda+\mu}$ *under* $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \to \mathcal{B}(\lambda+\mu)$ *and* η *is the indicator function of the Cartan component.*
2. $\mathbf{f}_i^\lambda \mathbf{v}_j^\mu = \sum_{k,l} \mathbf{v}_k^\mu \mathbf{f}_l^\lambda,$

   with sum over (k,l) *such that.* $\sigma : b_l^\lambda \otimes b_j^\mu \mapsto b_k^\mu \otimes b_i^\lambda.$
3. $\sum_i \mathbf{v}_i^\lambda \mathbf{f}_i^\lambda = 1.$
4. $(\mathbf{f}_i^\lambda)^* = \mathbf{v}_i^\lambda.$

Now put:

- $p_v = \prod_i \mathbf{v}_{b_i} \mathbf{f}_{b_i}$ for $v = (b_1, \ldots, b_r) \in \Lambda^0$,
- $s_{(v,b)} = \mathbf{v}_b p_v$ for $(v,b) \in \Lambda$.

$\Rightarrow$ KP(Λ) relations. □

Thank you.